* + - * 1. **Normal forms**

1. A formula is in disjunctive normal form (DNF), if it is written as a disjunction of cubes:

, where  are literals.

1. A formula is in conjunctive normal form (CNF), if it is written as a conjunction of clauses:

, where  are literals.

**Exercise 5**

Transform the formulas  into their equivalent conjunctive and disjunctive normal forms. Using one of these forms prove that  are valid formulas in propositional logic.

**U1=** ( **p -> Ꞁ q) ^(q v r) -> (p->r)**

**Logical connectives: V  ˄   Ꞁ   →    ↔**  **↑ ↓ **

**Meta-symbols (express binary semantic relations): |= ≡**

**Exercise 6**

Using the appropriate normal form write all the models of the following formulas:

**U1= (p v q -> r) -> (p->r) ^q**

**Logical connectives: V  ˄   Ꞁ   →    ↔**  **↑ ↓ **

**Meta-symbols (express binary semantic relations): |= ≡**

* + - * 1. **Theoretical results:**

1. **DNF provides the models of a formula**
2. **CNF/DNF are obtained by applying the normalization algorithm**
   * + - 1. **Apply the normalization algoritm:**

**U1= (p v q -> r) -> (p->r) ^q , replace -> using X->Y ≡ ꞀX v Y**

**≡ Ꞁ (p v q -> r) V (p->r) ^q , replace ->**

**≡ Ꞁ (Ꞁ (p v q) V r) V (Ꞁ p V r) ^q, de Morgan’s law**

**≡ ((p v q) ^ Ꞁ  r) V ( Ꞁ p V r) ^q, distributive laws**

**≡ (p ^ Ꞁ  r) v (q ^ Ꞁ  r) V ( Ꞁ p ^q) V (r ^q), It is an DNF with four cubes**

**Cube (p ^ Ꞁ  r) provides 2 models:**

* + - * 1. **i1, i2:{p,q,r}->{T,F},**
        2. **Cube (q ^ Ꞁ  r) provides 2 models:**
        3. **i3, i4:{p,q,r}->{T,F},**
        4. **i3(p)=T, i3(q)=T i3(r)=F , i3(q ^ Ꞁ  r) =T, i3(U1)=T**
        5. **i4(p)=F, i4(q)=T i4(r)=F , i4(q ^ Ꞁ  r))=T, i4(U1)=T**

**Cube ( Ꞁ p ^q) provides 2 models:**

* + - * 1. **i5, i6:{p,q,r}->{T,F}**
        2. **i5(p)=F, i5(q)=T i5(r)=T , i5((q ^ Ꞁ  r) =T, i5(U1)=T**
        3. **i6(p)=F, i6(q)=T i6(r)=F , i6((q ^ Ꞁ  r))=T, i6(U1)=T**

**Cube (r ^q) provides 2 models:**

* + - * 1. **i7, i8:{p,q,r}->{T,F}**
        2. **i7(p)=T, i7(q)=T i7(r)=T , i7((r ^q) =T, i7(U1)=T**
        3. **i8(p)=F, i8(q)=T i8(r)=T , i8((r ^q))=T, i8(U1)=T**
        4. **We notice that i4=i6 and i5=i8 and i1=i3**
        5. **Conclusion:**
        6. **The formula U1 has 5 models i1,i2, i4, i5, i7**
      1. **Exercise 7.**
      2. Using the appropriate normal form, prove that the following formulas are inconsistent:

**U1= (p-> (q->r))** ^ **Ꞁ ((p->q) ->(p->r))**

**Logical connectives: V  ˄   Ꞁ   →    ↔**  **↑ ↓ **

**Meta-symbols (express binary semantic relations): |= ≡**

**Exercise 8**

Write all the anti-models of the following formulas using CNF.

**U1= ( q^r->p) -> (p->r)^ q**

Theoretical result:

**Logical connectives: V  ˄   Ꞁ   →    ↔**  **↑ ↓ **

**Meta-symbols (express binary semantic relations): |= ≡**

U = (q ∧ r → p) → (p → r) ∧ q ,replace →

≡ ¬(q ∧ r → p) ∨ (p → r) ∧ q ,replace →

≡ ¬(¬(q ∧ r) ∨ p) ∨ (¬p ∨ r) ∧ q ,apply De Morgans laws

≡(q ∧ r ∧ ¬p) ∨ (¬p ∨ r) ∧ q ,apply distributivity law

≡(q ∧ r ∧ ¬p) ∨ (¬p ∧ q) ∨ (r ∧ q) ,we obtained DNF with 3 cubes

≡~~(q ∨ ¬p ∨ r)~~ ∧ ~~( q∨¬p ∨ q~~) ∧ ~~(q ∨ q ∨ r)~~ ∧ (q ∨ q ∨ q) ∧

(r ∨ ¬p ∨ r) ∧ ( ~~r ∨¬p ∨ q)~~ ∧ (~~r ∨ q ∨ r~~) ∧ ~~(r ∨ q ∨ q)~~ ∧

( ¬p ∨ ¬p ∨ r) ∧ (~~¬p ∨¬p ∨ q)~~ ∧ (~~¬p ∨ q ∨ r) ∧ (¬p ∨ q ∨ q)~~

CNF with 12 clauses

≡ q ∧ (¬p ∨ r) simplified CNF

Absorption laws: (Xv Y) ^X ≡X

(X^Y) v X ≡X

Clause q provides 4 anti-models:

I1,i2,i3,i4:{p,q,r}->{T,F}

* + - * 1. **i1(p)=T, i1(q)=F,, i1(r)=T , i1(q)=F, i1(U1)=F**
        2. **i2(p)=F, i2(q)=F,, i2(r)=F , i2(q)=F, i1(U1)=F**
        3. **i3(p)=T, i3(q)=F i3(r)=F , i3(q) =F, i3(U1)=F**
        4. **i4(p)=F, i4(q)=F i4(r)=T , i4(q )=F, i4(U1)=F**

Clause ¬p ∨ r) provides 2 anti-models:

* + - * 1. **i5, i6:{p,q,r}->{T,F}**
        2. **i5(p)=T, i5(q)=T i5(r)=F , i5(**¬p ∨ r**) =F, i5(U1)=F**
        3. **i6(p)=T, i6(q)=F i6(r)=F , i6(**¬p ∨ r**)=F, i6(U1)=F**
      1. **We deduce that: i3=i6**
      2. **In conculsion U1 has 5 anti-models and they are: i1,i2, i3, i4,i5**

**Exercise 9.**

Using the definition of deduction, prove the following deductions:

**p-> q, r->t, p v r, Ꞁq |- t**

**Logical connectives:          V  ˄   Ꞁ   →    ↔**  **↑ ↓  Å**

**Meta-symbols (express binary semantic relations): |= ≡**

**We build a sequence of formulas:**

**F1: p →q X→Y ≡ Ꞁ X V Y**

**F2: r →t**

**F3: p V r ≡ Ꞁp → r**

**F4:  Ꞁq**

**F4, F1 |-mt  Ꞁ p (apply modus tollens)**

**F5: Ꞁ p**

**F5, F3 |- mp r (apply modus ponens)**

**F6: r**

**F6, F2 |- mp t (apply modus ponens)**

**F7: t**

**The sequence of formulas (F1, F2, F3, F4, F5, F6, F7) is the proof (deduction) of t from the set of hypotheses (F1,F2,F3,F4)**

**Exercise 10:**

Prove the following theorems using the theorem of deduction and its reverse.

**|- p v (q->r) -> ((p v q) -> (p v r))**

**Logical connectives:          V  ˄   Ꞁ   →    ↔**  **↑ ↓  Å**

**Meta-symbols (express binary semantic relations): |= ≡**

**Exercise 11:**

Using the theorem of deduction and its reverse prove that:

**|- (p-> (q v r)) -> ((p-> q) v (p->r))**

**Logical connectives:          V  ˄   Ꞁ   →    ↔**  **↑ ↓  Å**

**Meta-symbols (express binary semantic relations): |= ≡**

**Exercise 12**

1. H1: It is not sunny this afternoon and it is colder than yesterday.
2. H2: We will go swimming only if it is sunny.
3. H3: If we do not go swimming, then we will take a canoe trip.
4. H4: If we take a canoe trip, then we will be home by sunset.

C: We will be home by sunset.

Is C deducible from the set of hypotheses {H1,H2,H3,H4}?

1. If yes, build its deduction.
2. Su – It is sunny
3. Co – It is colder
4. Sw – we will go swimming
5. Ct – we will take a canoe trip
6. Ss – we will be home by sunset
7. Propositional formulas:

H1: **Ꞁ** Su ^ Co

H2: Sw **→** Su

H3: **Ꞁ** Sw **→** Ct

H4: Ct **→** Ss

1. C: Ss

We build a sequence of formulas: (H1,H2,H3,H4,….)

H1 |- simplif **Ꞁ** Su :F5

F5, F2|- mt **Ꞁ** Sw :F6 (modus tollens)

F6, F3|- mp Ct: F7 (modus ponens)

F7, F4|- mp Ss: F8 = C (modus ponens)

The sequence of formulas (H1, H2, H3, H4, f5, f6, f7, f8) is the deduction of C from the hypotheses therefore, based on the hypotheses, we will be home by sunset.